On an identity for the volume integral of the square of a vector field. Remark on a paper by A. M. Stewart

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Gubarev, Stodolsky, and Zakharov[1] have noted the following identity for vector fields that vanish sufficiently rapidly at spatial infinity,

$$\int d^3x\, \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x} = \frac{1}{4\pi}\int d^3x\, d^3x' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big[\big(\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x})\big)\big(\boldsymbol{\nabla}'\cdot\boldsymbol{A}(\boldsymbol{x}')\big) + \big(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{\nabla}'\times\boldsymbol{A}(\boldsymbol{x}')\big)\Big], \tag{1}$$

and used it to investigate properties of the vector potential in quantum field theory. Stewart[2] has shown that the identity is also of general interest in classical electromagnetic theory. It can be used, for example, to derive easily interpreted expressions for the energies in time-dependent electric and magnetic fields.

The existence of this identity is not obvious. In Ref. 1 it is not proven, but it is noted that it follows from the momentum-space relation $(\boldsymbol{k}\times\tilde{\boldsymbol{A}})^2 = \boldsymbol{k}^2\tilde{\boldsymbol{A}}^2 - (\boldsymbol{k}\cdot\tilde{\boldsymbol{A}})^2$ where $\tilde{\boldsymbol{A}}(\boldsymbol{k})$ is the Fourier transform of the vector field. The position-space derivation given by Stewart is based on the Helmholtz decomposition of a three-dimensional vector field into irrotational and solenoidal parts. We give here an alternative position-space derivation that uses only familiar operations starting from the relations

$$\delta^3(\boldsymbol{x}-\boldsymbol{x}') = -\frac{1}{4\pi}\nabla^2\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|)} \tag{2}$$

for the Dirac delta function, and

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}. \tag{3}$$

Both are familiar to students and are used in the solution of Poisson's equation for the potential of a point charge and the derivation of the wave equation from Maxwell's equations.

By using Eq. (2), we obtain

$$\begin{aligned}\int d^3x\, \boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}) &= \int d^3x\, d^3x'\, \boldsymbol{A}(\boldsymbol{x})\cdot\delta^3(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{A}(\boldsymbol{x}')\\ &= -\frac{1}{4\pi}\int d^3x\, d^3x'\, \boldsymbol{A}(\boldsymbol{x})\cdot\Big(\nabla'^2\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big)\boldsymbol{A}(\boldsymbol{x}').\end{aligned} \tag{4}$$

If we integrate by parts over $\boldsymbol{x}'$ and use the assumed rapid vanishing of $\boldsymbol{A}(\boldsymbol{x}')$ for $|\boldsymbol{x}'|\to\infty$, we can transfer the action of the derivatives from the factor $1/|\boldsymbol{x}-\boldsymbol{x}'|$ to $\boldsymbol{A}(\boldsymbol{x}')$ without acquiring extra surface terms. We use the identity (3) for the double curl, make another

partial integration, and rewrite the result as

$$\int d^3x \boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}) = -\frac{1}{4\pi}\int d^3x\, d^3x' \boldsymbol{A}(\boldsymbol{x})\cdot\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\nabla'^2\boldsymbol{A}(\boldsymbol{x}') \tag{5a}$$

$$\begin{aligned} &= -\frac{1}{4\pi}\int d^3x\, d^3x' \boldsymbol{A}(\boldsymbol{x})\cdot\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big[\boldsymbol{\nabla}'\big(\boldsymbol{\nabla}'\cdot\boldsymbol{A}(\boldsymbol{x}')\big) - \boldsymbol{\nabla}'\times\big(\boldsymbol{\nabla}'\times\boldsymbol{A}(\boldsymbol{x}')\big)\Big] \\ &= \frac{1}{4\pi}\int d^3x\, d^3x'\Big[\Big(\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big)(\boldsymbol{\nabla}'\cdot\boldsymbol{A}(\boldsymbol{x}') \\ &\qquad - \boldsymbol{A}(\boldsymbol{x})\cdot\Big(\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big)\times\big(\boldsymbol{\nabla}'\times\boldsymbol{A}(\boldsymbol{x}')\big)\Big]. \end{aligned} \tag{5b}$$

Now $\boldsymbol{\nabla}'(1/|\boldsymbol{x}-\boldsymbol{x}'|) = -\boldsymbol{\nabla}(1/|\boldsymbol{x}-\boldsymbol{x}'|)$, so with some rearrangement of the vector products, Eq. (5b) becomes

$$\begin{aligned} \int d^3x \boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}) \;=\; &-\frac{1}{4\pi}\int d^3x\, d^3x'\Big[\big(\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\big)\big(\boldsymbol{\nabla}'\cdot\boldsymbol{A}(\boldsymbol{x}')\big) \\ &+\big(\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\big)\times\boldsymbol{A}(\boldsymbol{x})\cdot\big(\boldsymbol{\nabla}'\times\boldsymbol{A}(\boldsymbol{x}')\big)\Big]. \end{aligned} \tag{6}$$

A final partial integration with respect to $\boldsymbol{x}$ gives the desired identity.

The derivation generalizes easily to give an identity of the same form for the volume integral of the product of two different rapidly decreasing vector fields.

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[1] F. V. Gubarev, L. Stodolsky, and V. I. Zakarov, "On the significance of the vector potential squared," Phys. Rev. Lett. 86 (11), 2220–2222 (2001).

[2] A. M. Stewart, "On an identity for the volume integral of the square of a vector field," Am. J. Phys. (preceding paper).